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COMMENT

Characteristic function and Spitzer's law for the winding angle distribution of planar Brownian curves

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Abstract. Using the analogy between Brownian motion and quantum mechanics, we study the winding angle θ of planar Brownian curves around a given point, say the origin O. In particular, we compute the characteristic function for the probability distribution of θ and recover Spitzer's law in the limit of infinitely large times. Finally, we study the (large) change in the winding angle distribution when we add a repulsive potential at the origin.

Over several decades, topologically constrained Brownian curves have aroused a great interest among polymer physicists [1] and mathematicians [2]. For example, a large amount of work has been devoted to the study of winding properties of such curves [1-5]. In 1958, Spitzer [3] first calculated the asymptotic probability distribution for the winding angle θ of a planar Brownian curve around any point, O, different from the starting point. This result, in the limit of an infinitely large time τ, can be written:

$$P\left(X=\frac{2\theta}{\ln \tau}\right)=\frac{1}{\pi(1+X^2)}. \tag{1}$$

This is a Cauchy law. In particular, it has an infinite variance, a property first deduced by Levy [4].

In [5], Rudnick and Hu studied the influence of an excluded region, of radius ρ, enclosing the origin. In the limit of large times, they showed that $P(X)$, at large $|X|$, is an exponentially decreasing function of $|X|$ as long as $\rho \neq 0$. Thus, in that case, the variance is finite. More, they recovered Spitzer's law when $\rho \to 0$.

In this comment we compute the characteristic function for the winding angle distribution. This is done by solving a quantum mechanical problem, the topological constraint leading to the presence of a vortex field at the origin. The corresponding probability distribution takes a very simple form in the limit $\tau \to +\infty$ (Spitzer's law). We also show that this distribution is fundamentally changed when we introduce a repulsive potential ($\propto r^{-2}$) at the origin.

To begin, we shall consider the planar random walks starting at a fixed point $\boldsymbol{r}$ (polar coordinates (r, ϕ), $r \neq 0$) and ending after a time τ at $\boldsymbol{r}'((r', \phi+\theta)$, r' unspecified except for $r' \neq 0$, $(\phi+\theta)$ fixed). The probability to have a winding angle θ around O will be denoted $P(\theta)$. *A priori*, it depends on the starting point $\boldsymbol{r}$. However, we will see that this dependence is washed out when $\tau \to +\infty$.

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In the Wiener integral representation, $P(\theta)$ reads:

$$P(\theta)=N\int_{\varepsilon>0}^{+\infty}\mathrm{d}r'\int_{r(0)=r}^{r(\tau)=r'}[\mathscr{D}r]\delta\left(\int_0^\tau \dot\phi(t)\,\mathrm{d}t-\theta\right)\exp\left(-\frac{1}{2l}\int_0^\tau \dot r^2(t)\,\mathrm{d}t\right) \tag{2}$$

where N is a normalisation constant.

Equation (2) calls for two remarks:

(i) the measure is $\mathrm{d}r'$, and not $\mathrm{d}^2\boldsymbol{r}'$, the polar angle of $\boldsymbol{r}'$ being fixed;

(ii) the lower bound is $\varepsilon>0$ because $r'\neq 0$. However, we will see that taking $\varepsilon=0$ causes no trouble.

More, we shall take $l=1$ in the path integral (this choice of a unit length will not affect our final result).

Using the identity $2\pi\delta(x)=\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda$, (2) becomes:

$$\begin{aligned}P(\theta)&=\frac{N}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}\lambda\theta}\int_{\varepsilon}^{+\infty}\mathrm{d}r'\int_{r(0)=r}^{r(\tau)=r'}[\mathscr{D}r]\exp\left(-\int_0^\tau\frac{\dot r^2(t)}{2}-\mathrm{i}\lambda\dot\phi(t)\right)\mathrm{d}t\\&\equiv\frac{N}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}\lambda\theta}\int_{\varepsilon}^{+\infty}\mathrm{d}r'\langle\boldsymbol{r}'|\mathrm{e}^{-H\tau}|\boldsymbol{r}\rangle\end{aligned} \tag{3}$$

according to standard textbooks [6].

The Hamiltonian H, appearing in (3), is written

$$H=\frac{1}{2}\left(-\partial_r^2-\frac{1}{r}\partial_r+\frac{1}{r^2}(-\mathrm{i}\partial_\phi-\lambda)^2\right). \tag{4}$$

It describes a particle of unit charge moving in a vortex field localised at the origin and carrying a flux $2\pi\lambda$.

Practically, it will be more convenient to use a harmonic well regulator and consider the Hamiltonian H_ω:

$$H_\omega=H+\tfrac{1}{2}\omega^2r^2. \tag{5}$$

(We do not introduce a new notation for $P(\theta)$ although we change the Hamiltonian. We shall compute this probability in the limit $\omega\to 0$.)

The energy levels of H_ω are given by:

$$E_{M,p}=(|M-\lambda|+2p+1)\omega \tag{6}$$

(M and p integers, $p\geqslant 0$), the corresponding normalised eigenfunctions being:

$$\psi_{M,p}(\boldsymbol{r})=\frac{1}{\sqrt{A_{M,p}}}\mathrm{e}^{\mathrm{i}M\phi}\,\mathrm{e}^{-\omega r^2/2}r^{|M-\lambda|}L_p^{|M-\lambda|}(\omega r^2) \tag{7}$$

where

$$A_{M,p}=\pi\left(\frac{1}{\omega}\right)^{|M-\lambda|+1}\frac{\Gamma(p+|M-\lambda|+1)}{p!} \tag{8}$$

and $L_p^{|M-\lambda|}(\omega r^2)$ are Laguerre polynomials. Recall that [7]:

$$L_n^\alpha(X)=\Gamma(n+\alpha+1)\left(\sum_{K=0}^{\infty}\frac{(-1)^K X^K}{K!(n-K)!\Gamma(\alpha+K+1)}\right). \tag{9}$$

In that context, (3) can be rewritten as:

$$P(\theta)=\frac{N}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}\lambda\theta}\int_{\varepsilon}^{+\infty}\mathrm{d}r'\left(\sum_{M,p}\mathrm{e}^{-\tau E_{M,p}}\psi_{M,p}(\boldsymbol{r}')\psi^*_{M,p}(\boldsymbol{r})\right) \tag{10}$$

which gives, taking into account (6) and (7):

$$P(\theta) = \frac{N}{2\pi} e^{-\omega r^2/2} e^{-\tau\omega} \sum_M \left(\int_{-\infty}^{+\infty} d\lambda \, e^{i(M-\lambda)\theta} e^{-\tau\omega|M-\lambda|} r^{|M-\lambda|} \int_\varepsilon^{+\infty} F(r') \, dr' \right) \tag{11}$$

with

$$F(r') = \left(\sum_p \frac{e^{-2\tau\omega p} L_p^{|M-\lambda|}(\omega r^2) L_p^{|M-\lambda|}(\omega r'^2)}{A_{M,p}} \right) r'^{|M-\lambda|} e^{-\omega r'^2/2}$$
$$\equiv S(r') r'^{|M-\lambda|} e^{-\omega r'^2/2}.$$

($A_{M,p}$ is given by (8).)

Setting $(M-\lambda) \equiv u$, we easily see that the summation over M only amounts to a change in the normalisation factor, called now N'.

Defining $u' \equiv |u|$, $X \equiv \omega R^2$, $Y \equiv e^{-2\tau\omega}$ and using (8) and (9) and, also, the relationships [7]:

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) y^n = (1-y)^{-\alpha-1} \exp\left(-\frac{xy}{1-y} \right)$$
$$\frac{p!}{(p-k)!} Y^p = Y^k \frac{\partial^k}{\partial Y^k} (Y^p) \tag{12}$$

we can write:

$$S(r') = \frac{\omega^{u'+1}}{\pi} \sum_{K=0}^{\infty} \frac{(-XY)^K}{\Gamma(u'+K+1)K!} \frac{\partial^K}{\partial Y^K} \left[(1-Y)^{-u'-1} \exp\left(-\frac{\omega r'^2 Y}{1-Y} \right) \right]. \tag{13}$$

Integrating $F(r')$ over r' (with $\varepsilon = 0$), we are left with the expression:

$$P(\theta) = N' \int_{-\infty}^{+\infty} du \, e^{iu\theta} e^{-\tau\omega u'} r^{u'} (2\omega)^{(u'+1)/2} \Gamma\left(\frac{u'+1}{2} \right) T(u')$$
$$T(u') = \sum_{K=0}^{\infty} \frac{(-XY)^K}{\Gamma(u'+K+1)K!} \frac{\partial^K}{\partial Y^K} (1-Y^2)^{-(u'+1)/2}. \tag{14}$$

$1/\Gamma(u'+K+1)$ can be expressed as $(1/2i\pi) \int_c e^v v^{-u'-K-1} \, dv$. (In order to have an integrand that is monovalued, a cut is done, in the complex plane, along the negative real semi-axis; the contour C is any curve which begins at $-\infty$ under the cut, goes round the origin and ends at $-\infty$ above the cut). In particular, we can choose the contour C such that it makes sense to write:

$$\sum_{K=0}^{\infty} \frac{(-XY/v)K}{K!} \frac{\partial^K}{\partial Y^K} (1-Y^2)^{-((u'+1)/2)} = \left[1 - Y^2 \left(1 - \frac{X}{v} \right)^2 \right]^{-((u'+1)/2)}. \tag{15}$$

Taking the limit $\omega \to 0$ (and also, $\omega\tau \to 0$, because the particle must only 'feel' the vortex field when the regulator progressively disappears) and integrating over v, we get a confluent hypergeometric function of the first kind $F(a, 2a; x)$ with $a = (u'+1)/2$ and $x = -r^2/2\tau$. We can express this function in terms of a modified Bessel function $I_{u'/2}(-r^2/4\tau)$ and use the duplication formula for gamma functions [7]. Finally, collecting all the factors, we observe that the powers of ω cancel. We are left with the following simple expression for the probability distribution, $P(\theta)$:

$$P(\theta) = \frac{1}{2\pi I_0(r^2/4\tau)} \int_{-\infty}^{+\infty} du \, e^{iu\theta} I_{|u|/2}\left(\frac{r^2}{4\tau} \right). \tag{16}$$

Thus, the desired characteristic function is written

$$G(u) = \frac{I_{|u|/2}(r^2/4\tau)}{I_0(r^2/4\tau)}. \tag{17}$$

Equation (17) is the main result of our work and can be used, of course, to calculate $P(\theta)$ numerically. We also notice that (16) is close to the one obtained by Rudnick and Hu (equation (2.17) of [5]). Our expression, however, is somewhat simpler. Now, we consider the limit of infinitely large times.

Thinking of the expression of $I_\mu(X)$ at small X, we easily see that $G(u)$ is peaked at small u when the time τ takes large values. Introducing the variable $X \equiv 2\theta/\ln\tau$ and taking the limit $\tau \to +\infty$, we can write:

$$\begin{aligned} P(X) &\to \frac{\ln\tau}{4\pi}\int_{-\infty}^{+\infty} du\, e^{iu(X\ln\tau)/2}\left(\frac{r^2}{8\tau}\right)^{|u|/2} \\ &\xrightarrow[\tau\to+\infty]{} \frac{1}{\pi}\,\frac{1}{1+X^2} \end{aligned} \tag{18}$$

($\ln(r^2/8) \ll \ln\tau$: the dependence on the starting point disappears).

We have recovered the Cauchy law first deduced by Spitzer.

Now, we add a repulsive potential at the origin and replace the Hamiltonian of (4) by $H + D^2/2r^2$ (we shall consider $D > 0$). The energy levels (6) become

$$E'_{M,p} = (\sqrt{(M-\lambda)^2 + D^2} + 2p + 1)\omega \tag{19}$$

and, more generally, all the previous reasoning remains valid if we replace u' by $\sqrt{u^2 + D^2}$.

The new characteristic function is simply given by:

$$G'(u) = \frac{I_{\sqrt{u^2+D^2}/2}(r^2/4\tau)}{I_{D/2}(r^2/4\tau)} \tag{20}$$

and the probability distribution $P'(\theta)$ can be written:

$$P'(\theta) = \frac{D}{2\pi I_{D/2}(r^2/4\tau)}\int_{-\infty}^{+\infty} d\varphi\, \cosh\varphi\, e^{iD\sinh\varphi\,\theta} I_{(D\cosh\varphi)/2}\left(\frac{r^2}{4\tau}\right). \tag{21}$$

In the limit of large τ, we get:

$$P'(\theta) = \frac{1}{2\pi} D\left(\frac{8\tau}{r^2}\right)^{D/2} \frac{\ln 8\tau/r^2}{\sqrt{(\frac{1}{2}\ln 8\tau/r^2)^2 + \theta^2}} K_1\left(D\sqrt{\left(\frac{1}{2}\ln\frac{8\tau}{r^2}\right)^2 + \theta^2}\right) \tag{22}$$

(K_1 is a modified Bessel function. When $D \to 0$, $K_1(D\sqrt{\ldots}) \sim 1/D\sqrt{\ldots}$, (22) gives back Spitzer's law).

Now, we consider the limit $|X| \to \infty$. Using $K_1(x) \sim e^{-x}/\sqrt{x}$ when $x \to +\infty$, we easily see that

$$P'(X) \propto \frac{\exp(-(D\ln\tau/2)|X|)}{|X|^{3/2}} \tag{23}$$

for large X and τ. So, the asymptotic distribution is vastly changed when we add a repulsive potential (even if D is very small).

In particular, it acquires a finite variance [5]: we can say that the short distance behaviour has been 'regularised' by the repulsive potential. We finally notice that, in contrast with [5], (23) depends on the quantity D characterising the repulsive potential. The probability distribution is, of course, determined by the concrete shape of the potential. The one we have used in the last part of this work is very simply tractable.

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